A note on the averaged null energy condition in quantum field theory

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Abstract

Locally, the stress-energy of quantized matter can become arbitrarily neg-
ative in a wide class of quantum states, thereby violating the classical
positive-energy conditions of General Relativity without bound. Since
without such constraints the theory would have no predictive power, un-
covering what nonlocal constraints, if any, quantum field theory imposes
on the renormalized stress-energy tensor is of central importance for semi-
classical gravity. One such nonlocal constraint, the averaged null en-
ergy condition (ANEC---the condition that the null-null component of the
stress-energy tensor integrated along a complete null geodesic is nonneg-
ative in every quantum state) bass been recently shown to hold for linear
quantum fields in a large elms of spacetimes. Nevertheless, it is easy
to show by using a simple scaling argument that ANEC as stated cannot
hold generically in curved four-dimensional spacetime, and this scaling ar-
gument has been widely interpreted as a death-blow for averaged energy

conditions in quantum field theory. In this note I propose a simple generalization of ANEC, in which the right-hand-side of the ANEC inequality is replaced by a finite (but in general negative) state-independent lower bound. As long as attention is focused on asymptotically well-behaved space times, this generalized version of ANEC is safe from the threat of the scaling argument, and thus stands a chance of being generally valid in four-dimensional curved spacetime. I argue that when generalized ANEC holds, it has implications for the non-negativity of total energy and for singularity theorems similar to the implications of AN EC. In particular, I show that if generalized ANEC is satisfied in static traversable wormhole spacetimes (which is likely but remains to be shown), then macroscopic wormholes (but not necessarily microscopic, Planck-size wormholes) are ruled out by quantum field theory.

The most striking aspect of the violation of positive-crlcr,gy conditions by quantum stress-energy tensors is the unbounded extent of the violation. For example, even for a Klein-Gordon scalar field in flat, Minkowski space, the regularized (normal-ordered) expectation value $\langle\omega|:T_{00}(x):|\omega\rangle$ at any point x is unbounded from below as a functional of the quantum state ω. Furthermore, the volume integral of $\langle\omega|:T_{00}(x):|\omega\rangle$ over any fixed, spacelike 3-box of *finite* size is also unbounded from below as a functional of ω (and a similar result holds for the spacetilnc-volume integral over a compact 4- box; see Sect. 1 and Ref. []] of [1] for more details). Given this tendency of the regularized expectation value $\langle\omega|T_{ab}(x)|\omega\rangle$ to become unboundedly negative, any condition that sets a lower bound on nonlocal averages of $\langle\omega|T_{ab}(x)|\omega\rangle$ would be a significant constraint on the quantum stress-energy tensor.

in this note I propose the following constraint as a generalization of the averaged null energy condition (A NEC; see [2] and [1] for a discussion of ANEC and its brief history): Let $\langle\omega|T_{ab}|\omega\rangle$ denote the (renormalized) stress-energy tensor of a quantum field on a curved spacetime $(\mathcal{M}, g)$, arid let y **c** $\mathcal{M}$ be a complete null geodesic, For k^a a given (parallel-propagated) tangent vector along -y, let me introduce the following quantity $\beta(k)$:

$$\beta(k) \equiv \inf_{\omega} \int_{\gamma} \langle\omega|T_{ab}|\omega\rangle k^a k^b \ dv \tag{1}$$

I will say that $\langle T_{ab}\rangle$ satisfies generalized ANEC along γ if $\beta(k) > -\infty$. Here the infimum is taken over all Hadamard states w of the quantum field, and the integral along γ is with respect to the affine parameter v which corresponds to the tangent vector k^a (i.e., $d\gamma^a/dv = k^a$). Properly interpreted, the quantity β is a 1-form, whose contraction with the tangent vector k^a is given by the right hand side of Eq. (1). More precisely, β is an element of the quotient space $T_p^*\mathcal{M}/N_p$, where p is a point on $\gamma, N_p \subset T_p^*\mathcal{M}$ is the subspace of all 1-forms $\alpha \in T_p^*\mathcal{M}$ which annihilate the tangent vector $\gamma_*[\alpha(\gamma_*) = O]$, and β is parallel transported along 7 so that it does not matter at which $p \in 7$ the quantity $\beta(\gamma_*)$ is evaluated.

The usual ANEC along 7 is recovered by setting $\beta(\gamma_*) \geqslant 0$. If the integrand on the right hand side of Eq. (1) is non-integrable for some Hadamard states w, a more precise version of generalized AN EC needs to be employed just like the more precise version of ANEC discussed in Sect. 2 of [2]. Namely, let $c(x)$ be a compact-supported real-valued function on $\mathbb{R}$ whose Fourier transform $\hat{c}(s)$ is such that for some $\delta > 0$ the function $(1 + s^2)^{1+\delta}|\hat{c}(s)|$ is bounded [which implies that $c(x)$ is C^1]. Generalized ANEC holds along a complete null geodesic 7 if for every such weighting function $c(x)$ the 1-form β_c along 7 defined by

$$\beta_c(k) \equiv \inf_{\omega} \liminf_{\lambda \to \infty} \int_{\gamma} \langle\omega|T_{ab}|\omega\rangle k^a k^b \, [c(v/\lambda)]^2 \, dv \tag{2}$$

satisfies $\beta_c(k) > -co$. Generalized ANEC reduces (or, more accurately, is strengthened) to ANEC when one imposes the stronger condition: $\beta_c(\gamma_*) \geqslant O$

for all weighting functions $c(x)$ as above. Note that generalized ANEC can be formulated equivalently in the (perhaps more sensible) form of an inequality: namely, for all $c(x)$ chosen as above,

$$\liminf_{\lambda \to \infty} \int_\gamma \langle \omega | T_{ab} | \omega \rangle k^a k^b \, [c(v/\lambda)]^2 \, dv \;\geqslant\; \beta_c(k) \qquad \forall \omega \;, \tag{3}$$

where the expression $\beta_c(k)$ on the right hand side is a state-independent lower bound for the (weighted) ANEC integral on the left hand side. In general (when generalized ANEC holds), this lower bound (i.e., the precise expression of the 1-form β_c) will depend only on the geometry of the spacetime $(\mathcal{M}, g)$ [as well, of course, as on the null geodesic γ and the weighting function $c(x)$]. If the infimum over ω in Eq. ('2) is achieved [for all c(z)] by some Hadamard state ω_0, so that

$$\beta_c(k) = \liminf_{\lambda \to \infty} \int_\gamma \langle \omega_0 | T_{ab} | \omega_0 \rangle k^a k^l \, [c(v/\lambda)]^2 \, dv \;,$$

then Eq. (3) can be written in the form of a *difference inequality* (see [3] and [I] on difference inequalities):

$$\liminf_{\lambda = \infty} \int_\gamma (\; \langle \omega | T_{ab} | \omega \rangle - \langle \omega_0 | T_{ab} | \omega_0 \rangle) \, k^a k^b \, [c(v/\lambda)]^2 \; dv \;\geqslant\; 0 \qquad \forall \omega \tag{4}$$

Conversely, if $\langle \omega | T_{ab}(x) | \omega \rangle$ satisfies a difference inequality

$$\liminf_{\lambda \to \infty} \int_\gamma (\, \langle \omega | T_{ab} | \omega \rangle - D_{ab}) \, k^a k^b \, [c(v/\lambda)]^2 \, dv \;\geqslant\; 0 \qquad \forall \omega$$

such that the expression $D_{ab} k^a k^b$ *is* integrable along γ, then generalized ANEC [Eq. (3)] holds with $\beta_c(k) \geqslant [c(O)]' \int_\gamma D_{ab} k^a k^b \, dv$.

Before 1 discuss the physical significance of generalized AN EC, let me explain why this modified version of ANEC has a better chance of holding generally in curved four-dimensional spacetime thau the original version. Recall the scaling argument given in the note added in proof to [2] and discussed in more detail recently in [4]: Restrict attention, for simplicity, to a massless Klein-Gordon scalar quantum field ϕ. Given an arbitrary four-dimensional spacetime $(\mathcal{M}, g)$ in which ϕ satisfies ANEC along a null geodesic γ, the scaling argument asks us to consider the new spacetime $(M, \kappa^2 g)$, where $\kappa > 0$ is a constant scale factor (in particular, in this new spacetime the curve -y $\subset \mathcal{M}$ is still a null geodesic with the same affine parameter v). 'To every Hadamard state ω of ϕ with two-point function $\mu_\omega(x, x')$ on the original spacetime, there! corresponds a Hadamard state ω on the scaled spacetime with two-point function $\kappa^{-2} \mu_\omega(x, x')$. [The massless Klein-Gordon equation is invariant under scale transformations (whereas the massive Klein-Gordon equation is scale-invariant upto a rescaling of the mass); therefore the function $\mu_\omega(x, x')$ remains a bi-solution of the massless Klein-Gordon equation under the scaling $g \to \kappa^2 g$. The overall scale factor κ^{-2} is introduced to keep μ_ω in Hadamard form in the new spacetime.] Normally, then, one would expect the regularized expectation value $\langle \omega | T_{ab} | \omega \rangle$ to simply scale as K^{-4} (because its definition involves differentiating the two-point function twice with respect to locally inertial coordinates). However, according to the general renormalization prescription for μ_ω, *before* the differentiations and

the limit $x \to x'$ are carried out to evaluate $\langle \omega | T_{ab}(x) | \omega \rangle$, a locally constructed Hadamard distribution $\mu_0(x, x')$ needs to be subtracted from $\mu_\omega(x, x')$ to obtain the regularized two-point function. It turns out that this local Hadamard distribution μ_0 does not scale in the same simple way as μ_ω under the scaling $g \to \kappa^2 g$ of the metric, and this anomalous scaling behavior of $\mu_0(x, x')$ ends up contributing two additional terms (apart from the simply scaled term $\kappa^{-4} \langle \omega | T_{ab} | \omega \rangle$) to the value of $\langle \omega | T_{ab} | \omega \rangle$ in the scaled spacetime ([5]). These additional terms are of the form $a \kappa^{-4} \ln \kappa \, ^{(1)}H_{ab}$ -1 $b \kappa^{-4} \ln \kappa \, ^{(2)}H_{ab}$, Where a, b are dimensionless (in Planck units) constants which have known universal values for each fixed quantum field, and $^{(1)}H_{ab}$ and $^{(2)}H_{ab}$ denote the conserved local curvature terms

$$^{(1)}H_{ab} \equiv 2R_{;ab} + 2RR_{ab} - g_{ab}(2 \Box R + \tfrac{1}{2}R^2) , \tag{5}$$

and

$$^{(2)}H_{ab} \equiv R_{;ab} - \Box R_{ab} + 2R_a{}^c R_{cb} - \tfrac{1}{2}g_{ab}(\Box R + R^{cd}R_{cd}) , \tag{6}$$

respectively. The precise numerical values of the constants a and b depend only on the spin and internal structure of the specific quantum field considered. [Note that although this scaling behavior of $(\omega | T_{ab} | \omega \rangle$ is closely related to the famous ambiguity in the renormalization prescription, the constants a and b are determined independently of this ambiguity. For most fields of interest their values can be found in the literature (see [5], p. 1450 for a table of these constants for various quantum fields; notice, however, that a and b in that table arc given with respect to different conserved curvature terms which arc linear combinations of $^{(1)}H_{ab}$ and $^{(2)}H_{ab}$). I will not need to specify the exact values of a and b in this note: it will suffice to know only the fact that in general these are constants with absolute magnitudes of order 10^{-4} (in Planck units).] It is now clear that if the curvature of the original spacetime is sufficiently general So that the integrals $\int_\gamma {}^{(1)}H_{ab}k^a k^b \, dv$ and $\int_\gamma {}^{(2)}H_{ab}k^a k^b \, dv$ are non-vanishing, then by choosing the scale factor κ appropriately (note that the logarithm $\ln \kappa$ has indefinite sign) it should be possible to find a spacetime $(\mathcal{M}, \kappa^2 g)$ in which ANEC is violated along γ. Notice, however, the crucial feature of the ANEC-violating term (proportional to $\kappa^{-4} \ln \kappa$) disclosed by this scaling argument: it is independent of the quantum state ω. Therefore, if as a functional of the quantum state the ANEC integral along γ is bounded from below in the original spacetime (as would be the case if A NEC holds there), with the greatest lower bound given by a 1-form β as in Eq. (1), then the only effect of the scaling $g \to \kappa^2 g$ will be to shift this lower bound β down (or up) by an amount proportional to $\kappa^{-4} \ln \kappa$ and the integrals of $^{(1)}H_{ab}k^a k^b$ and $^{(2)}H_{ab}k^a k^b$ along γ. When the spacetime $(\mathcal{M}, g)$ is asymptotically well-behaved (so that its Ricci curvature falls off appropriately at null infinity), these integrals are finite. Consequently, *if generalized A NEC holds along γ in the* asymptotically *flat (more precisely, asymptotically empty) spacetime $(\mathcal{M}, g)$, then it holds in* the scaled *spacetime $(\mathcal{M}, \kappa^2 g)$ for any $\kappa > 0$.*

in the remainder of this note I will argue that generalized ANEC, although a much weaker constraint than the usual ANEC, has physical significance quite similar to that of ANEC in semiclassical gravity. I will make this

argument by discussing in turn the implications of generalized ANEC for positivity of total energy, for singularity theorems, and for the existence of static traversable wormhole solutions to the semiclassical Einstein equations.

Positivity of total energy. - Recall the argument in Ref. [1] leading to the Theorem in Sect. 1 there. Instead of a T_{ab} satisfying ANEC in the simple form Eq. (3) of [1], consider a quantum stress-energy tensor (w $|T_{ab}|\omega\rangle$ which satisfies generalized ANEC in the form

$$\int_\gamma \langle \omega|T_{ab}|\omega\rangle k^a k^b \, dv \geq \beta_\gamma(k) \qquad \forall \omega \tag{7}$$

along all complete null geodesics γ in $(\mathcal{M}, g)$ [assume, in other words, that $\langle \omega|T_{ab}|\omega\rangle$ is integrable along each complete γ and satisfies Eq. (3) above]. Consider a Cauchy surface Σ and a subregion S C Σ as in [I], and modify the assumption (Al) in Sect. 1 of [I] to:

(A1) For each fixed Hadamard state w, let the subregion S C Σ be chosen large enough such that generalized ANEC[Eq. (7)] holds for $\langle \omega|T_{ab}|\omega\rangle$ along all null generators of the future horizon H^+ (S).

Assume also that the assumption (A2) holds as described in [I]. Then, using exactly the same arguments as in the proof of the Theorem in Sect. 1 of [1] [between Eqs.(6) and (10) there], it follows that either the total energy contained in S, $\int_S \langle \omega|T_{ab}|\omega\rangle n^a n^b \, d^3\sigma$, is nonnegative, or, if this total energy is negative, then it is bounded from below by a lower bound which depends on w only weakly through the choice of S. More precisely, it follows that

$$\int_S \langle \omega|T_{ab}|\omega\rangle n^a n^b \, d^3\sigma \geq$$
$$\min\left[0, \sup_\alpha\left(\left|\frac{1}{\|\nabla\alpha\|_s}\right|\int_{\gamma\in H^+(S)} d^2\Omega\,\beta_\gamma(-\nabla\alpha)\right)\right] \qquad \forall \omega, \tag{8}$$

where the supremum on the right hand side is over all time functions α which satisfy (for some constants κ, g > O) the conditions of assumption (A2), and the integral inside the Sup$_s$ is over all null generators of tile horizon H^+ (S), evaluated with respect to the unique "solid angle" -measure $d^2\Omega$ on the set of generators such that $d^2\Omega\,dv = d^3\sigma$ [where v is the affine parameter along the generators, and $d^3\sigma$ is the canonical volume element of H^+ (s)]. Consequently, just as ANEC places a positivity constraint on the integrated energy density under appropriate assumptions, so also generalized ANEC places, under similar assumptions, an essentially state-independent (in general negative) lower bound on the same quantity. (Note that in general the quantum state w determines exactly how "large" the region S C Σ needs to be chosen, and this is the only reason the lower bound might depend on ω.)

Singularity theorems--- To illustrate the relevance of generalized ANEC for singularity theorems and other global results of classical General Relativity, recall the Proposition proved in Sect. 2 of [2], which uses the constraint on the Ricci tensor imposed by ANEC and the Einstein equations to demonstrate a focusing lemma for null geodesics; a result of the kind which constitute

the key ingredient in the proof of global results such as singularity theorems. A straightforward reworking of the argument in the proof described in Ref. 2 [given between Eqs. (4) and (9) there] directly demonstrates the following variation of that Proposition:

Proposition: Let p be a point on a complete null geodesic $\gamma(v)$. Assume that γ satisfies the following property: For the specific choice of the weighting function $c(x)$ [see the formulation Eq. (3) of generalized ANEC above] given by

$$c(x) = c_1(x) = (1 - x^2)^2, |x| < 1 \ ,$$
$$c_1(x) = 0, \qquad\qquad |x| \geqslant 1 \ , \qquad\qquad (9)$$

the weighted average of the Ricci tensor R_{ab} along γ obeys the inequality

$$\liminf_{\lambda \to \infty} \int_0^\infty R_{ab} k^a k^b \, [c_1(v/\lambda)]^2 \, dv \ \geqslant \ \beta_1(k) \ , \qquad\qquad \textbf{(lo)}$$

where $\beta_1(k) > -\infty$ (and where $v = 0$ at p). [If generalized ANEC (together with semiclassical Einstein's equations) holds, then with $\beta_1 \equiv 4\pi\beta_{c_1}$ this condition must hold in every Hadamard state of the quantum field for at least one direction along γ from p.] Consider a null geodesic congruence containing γ whose expansion $\theta(v)$ along γ satisfies, initially at the point p, tile inequality

$$u(o) \leqslant \beta_1(k) \qquad\qquad (11)$$

(note that under a resealing of the affine parameter the expansion O scales in the same way as the tangent vector k, so this inequality is independent of the choice of affine parameter). 'l'hen, either θ vanishes identically along γ, or there exists a finite $v_0 > 0$ at which $\lim_{v \to v_0} \theta(v) = -\infty$.

Therefore, independently of which quantum state the field is in, generalized ANEC guarantees the refocusing of a null geodesic congruence if the initial convergence is sufficiently nonpositive, or, in other words (assuming $\beta_1 \leqslant$ O), if the initial convergence is more negative than the amount of ANEC violation allowed by generalized ANEC. As was also the case with ANEC (see the last paragraph on p. 405 of Ref. [2]), a proof can also probably be given that if generalized ANEC holds along a complete null geodesic γ, and if γ satisfies the null generic condition such that the maximum magnitude of the quantity $k^a k^b k_{[c} R_{d]ab[e} k_{f]}$ (which enters the formulation of the generic condition) is sufficiently large compared to the magnitude of ANEC violation allowed by generalized ANEC [cf. Eq. (11)], then γ must contain a pair of conjugate points.

Traversable wormholes. - A widely applicable generalized ANEC theorem would place a significant a priori constraint on possible solutions to the semiclassical Einstein equations. Namely, assume that such a theorem- to the effect that generalized ANEC holds along (cert sin) complete null geodesics γ in every asymptotically empty spacetime, with a geometric, state-independent lower bound /3Y---were available. Then, given auy spacetime $(\mathcal{M}, g)$, one could compute for each specified $\gamma \subset \mathcal{M}$ the quantity β_γ in the geometry of $(\mathcal{M}, g)$, and compare the result with the ANEC integral of the Einstein tensor $\frac{1}{8\pi} G_{ab}$ along the same null geodesic. If the comparison fails to satisfy the generalized ANEC

inequality [Eq. (3)] for at least some γ, then there cannot, exist any Hadamard quantum state which would make $(\mathcal{M}, g)$ a self-consistcut semiclassical solution of the Einstein equations; in other words, the spacetime $(\mathcal{M}, g)$ would be ruled out by quantum field theory (at least with the specific quantum fields for which a detailed analysis of the ANEC integrals can be carried out). A nice illustration of these ideas is provided by static (spherically symmetric) traversable wormhole spacetimes. Such a wormhole has topology $S^2 \times \mathbb{R}^2$, and a metric of the general form

$$g = -e^{2\Phi(l)} dt^2 + dl^2 + r(l)^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) , \qquad (12)$$

where the radial coordinate 1 ranges from $-\infty$ (on one asymptotic region) to $+\infty$ (on the other). For there to be no event horizons (hence for the wormhole to be traversable), Φ needs to be finite everywhere. For asymptotic flatness, it is necessary that as $l \to \pm \infty$ (more precisely, for $|l| \gg r_0$)

$$r(l) \simeq |l| - M \ln \left(\frac{|l|}{r_0} \right) , \text{ and } \Phi(l) \simeq -\frac{M}{|l|} , \qquad (13)$$

where r_0 is the radius of the wormhole's "throat," (where 1 varnishes), and M is the wormhole's mass. Throughout my discussion here 1 will assume that $r_0 \sim 2M$ (which should be the case if as seen from infinity the wormhole is indistinguishable from an astrophysical object); as a result, the class of worm-holes 1 will consider is parametrized (essentially) by one variable: the wormhole mass M. For more details on wormholes see the discussion in [6]; for a more uI--to-date account (including a discussion of the more recent work on ANEC) see [7]

The spacetime given by Eqs. (12-13) [with everywhere regular $\Phi(l)$] violates ANEC along all its radial null geodesics: a straightforward computation of the Einstein tensor followed by an integration by parts reveals that

$$V \equiv \frac{1}{8\pi} \int_\gamma G_{ab} k^a k^b \, dv = -\frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-\Phi} \left(\frac{r'}{r} \right)^2 dl \qquad (14)$$

along any radial null geodesic γ (note that the affine parameter v along γ can be chosen to be any positive constant times $\int e^\Phi dl$; 1 will choose this constant to be unity throughout so that $dv = e^\Phi dl$ and $k = e^{-2\Phi} \partial/\partial l + e^{-\Phi} \partial/\partial l$). Here and in what follows a prime (') denotes differentiation with respect to the radial coordinate 1. Substituting Eq. (13) in the last integral of Eq. (14) (and carrying out the integration only over $|l| \geqslant 2M$) gives

$$V \approx \frac{67}{384\pi} \frac{1}{M} \qquad (15)$$

Can this ANEC violation V necessary to maintain a traversable wormhole be supported by a quantum stress-energy tensor? Consider a massless Klein-Gordon field on the wormhole spacetime (the answer is not likely to depend significantly on the spin or internal structure of the field). Assume that generalized ANEC holds along the radial null geodesics of the wormhole. Whether or not this assumption is true remains to be shown; however, the scaling argument

sketched above combined with the known ANEC theorems ([2]) in two and four dimensions suggest that it is likely to be true (note that the wormhole's radial null geodesics are complete and achronal). Proceeding with the assumption that generalized ANEC holds, how can we guess the form that the (finite) ANEC lower bound $\beta(k)$ is likely to take along the radial null geodesics of the wormhole spat-cti]nc? One way to approach this question is to look closely at the scaling behavior of the wormhole metric Eq. (12). in general, a scaling $g \to \kappa^2 g$ of Eq. (12) leads to a new wormhole metric $g = \kappa^2 g$ for which the metric functions Φ and r arc given by

$$\bar{\Phi}(l) = \Phi(l/\kappa), \quad \text{and} \quad \bar{r}(l) = \kappa\, r(l/\kappa), \tag{16}$$

where a har over a symbol indicates that the corresponding quantity refers to the scaled spacetime with metric $g = \kappa^2 g$. From Eq. (14) it follows quite generally that $V = \kappa^{-1}V$, and from Eq. (13) it follows that $\bar{r}_0 = \kappa r_0$ and [consistent with Eq. (15)] $M = \kappa M$. From the scaling argument 1 described above for a general spacetime, it follows that

$$\overline{\beta(k)} = \frac{1}{\kappa^3}\beta(k) + \frac{\ln\kappa}{\kappa^3}\int_\gamma \left(a\,^{(1)}H_{ab}k^a k^b + b\,^{(2)}H_{ab}k^a k^b\right)\,dv, \tag{17}$$

where $^{(1)}H_{ab}$ and $^{(2)}H_{ab}$ arc the conserved curvature terms given by Eqs. (5) and (6), respectively. Computer algebra systems ([8]) make the computation of these higher curvature terms easier: for the wormhole metric [Eq. (12)] 1 find

$$
\begin{aligned}
^{(1)}H_{ab}k^a k^b \; dv \;=\; & -8\;\mathrm{TD} - \left(e^{-\Phi}\right)''\Phi'' - 3\left(r'e^{-\Phi}\right)'\frac{r'\Phi'}{r^2} \\
& + e^{-\Phi}\left(\Phi'''' -,- \frac{r'}{r}\Phi'\Phi'' - \frac{(r')^2}{r^2}\Phi'' + \frac{r'}{r}(\Phi')^3\right. \\
& -t\,3\frac{(r')^2}{r^4}[1 -t\,(?\;'')2] + \frac{r''''}{r} - \frac{(r'')^2}{r^2} - 6\frac{r''}{r^3}(r')^2 \\
& \left. \frac{r''}{r^2}r'\Phi' + 4\frac{(r')^3}{r^3}\Phi - 2\frac{r''}{r}(\Phi')^2\right]dl, \tag{18}
\end{aligned}
$$

and

$$
\begin{aligned}
^{(2)}H_{ab}k^a k^b \; dv \;=\; & -2\;\mathrm{TD} - 2\left(e^{-\Phi}\right)''\Phi'' - 7\,(r'e^{-\Phi})'\frac{r'\Phi'}{r^2} \\
& + e^{-\Phi}\left(\Phi'''' + 5\frac{r'}{r}\Phi'\Phi'' + \frac{r'}{r}\Phi''' + !::@\right. \\
& + 2\frac{(r')^2}{r^4}[1 -t\,2(r')^2] -2\frac{(r'')^2}{r^2} - 6\frac{r''}{r^3}(r')^2 \\
& \left. + 4\frac{(r')^3}{r^3}\Phi' - \frac{r''}{r}(\Phi')^{``}\right)ld/, \tag{19}
\end{aligned}
$$

where the symbols TD denote total derivative terms of the form $dF[\Phi,r]/dl$ with $\lim_{l\to\pm\infty}F' = 0$ under the boundary conditions Eqs. (1 3). A calculation similar to the derivation of Eq. (15) from Eq. (14) gives, when applied to Eqs. (18)- (19),

$$\int_\gamma {}^{(1)}H_{ab}k^a k^b \; dv \approx \frac{3}{128}\frac{-1221 + 20048\ln(2)}{M^3} \sim 46\frac{1}{M^{3}}, \tag{20}$$

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and

$$\int_\gamma {}^{(2)}H_{a\ b}k^a k^b dv \approx \frac{1}{2520} \cdot \frac{-28117 + 47040\ln(2)}{M^3} \approx 1.78\frac{1}{M^3} \qquad (21)$$

Now let $B(M)$ denote the value of the quantity $\beta(k)$ for a wormhole spacetime [Eqs. (12) (13)] of mass M (recall $r_0 \sim 2M$). Note that, for simplicity, 1 have been ignoring' in this discussion the more accurate version Eqs. ('2)- (3) of generalized ANEC; more precisely, 1 have assumed (and will continue to assume) that for a general weighting function $c(x)$ it holds that $\beta_c(k) = [c(0)]^2 \beta(k)$. Combining Eqs. (20)- (21) with Eq. (17) and recalling that $a \sim b \sim 10^{-4}$, 1 deduce the relation

$$B(\kappa M) = \frac{1}{\kappa^3}B(M) + \frac{\ln \kappa}{\kappa^3}\frac{10^{-4}c}{M^3}, \qquad (22)$$

where c is a numerical constant with $|c| \sim 1$. It is reasonable to guess (and this is the only "guessing" involved in the present argument) that $|B(1)| \sim 1$; i.e., that the value of $B(M)$ for a Planck-mass $(M \sim 1)$ wormhole is (in absolute magnitude) of order unity, that is, of Planck size (in Planck units). Then lq. (22) gives

$$B(M) \approx \frac{1}{M^3}\left(c_1 + 10^{-4}c_2 \ln M\right), \qquad (23)$$

where c_1 and c_2 are constants with $|c_1| \sim |c_2| \sim 1$. Clearly, for reasonable M the second term in parenthesis in Eq. (23) is negligible compared to the first; hence $|B(M)| \sim 1/M^3$. In order to have the ANEC violation V [Eq. (15)] supportable by the renormalized stress-energy tensor, it is necessary t hat $B(M) \lesssim V$, which is only possible if $|B(M)| \gtrsim |V|$, which implies $M^2 \lesssim C_3$ where $c_3 \sim 1$. Therefore, *if generalized A NEC holds, a quantum Klein- Gordon field can* support the A NEC violation necessary for a traversable *wormhole only if the wormhole has Planck* mass or less; *in other words, all traversable wormholes except possibly those of Planck size are ruled out by quantum field* theory.

It is important to keep in mind that this conclusion rests entirely on the assumption that generalized ANEC holds along the radial null geodesics of the wormhole spacetime. If it can be shown that the ANEC integral along these geodesics is unbounded from below as a functional of the quantum state, then no amount of ANEC violation can be ruled out by quantum field theory; in particular, there might exist states in whit.b tile violation Eq. (15) necessary for a macroscopic traversable wormhole is supported by the expected quantum stress-energy tensor. Even if generalized ANEC could be shown to hold by indirect methods, a rigorous computation of the lower bound $\beta(k)$ will be necessary to demonstrate that it indeed has the behavior described in Eqs. (22)- (23). [It is plausible to conjecture that if generalized ANEC holds, then it can be put in the form Eq. (4), with the "minimum-ANEC- integral" state ω_0 being given by the standard, isometry-invariant vacuum for the wormhole metric.] Also, my conclusion is based on the analysis of a simple one-parameter family of wormholes; the general criterion to decide which wormhole metrics Eq. (12) are allowed in semiclassical gravity is bound to be much more complicated. For example, one could c11oosc the function $r(l)$ to be very slowly varying so as to make the integral in Eq. (14) as small (in absolute value) as the microscopic

A N EC violation allowed by quantum field theory. It appears that a macroscopic wormhole of this kind (with extremely small radial curvature) could be allowed even when generalized ANEC holds; however, such a spacetime would look more like a constant-radius $S^2 \times \mathbb{R}^2$ universe than a wormhole joining two asymptotically flat regions. Finally, although the main conclusion is almost certainly independent of the spin and internal structure of the specific Klein-Gordon field, the argument above deals only with the question of maintaining a wormhole using free (non-interacting) quantum fields. A plausible configuration for wormhole maintenance based on the Casimir effect (where an electromagnetic field providing the negative Casimir energy interacts with the matter fields in the conducting plates which trap it in the Casimir vacuum state) was outlined in Ref. [6]. To rule out macroscopic configurations of that sort, one would need a generalized ANEC theorem applicable to interacting fields. Nevertheless, it is difficult to see how interactions could induce a violation of generalized A N EC if such a theorem holds generally in non-interacting quantum field theory.

Similar conclusions about the constraint imposed by generalized ANEC on solutions of semiclassical Einstein equations can be reached for spacetimes more general than wormholes. For example, in Ref. [9] it is shown that when the classical null energy condition holds, a region with non-trivial topological structure cannot be visible from infinity in an asymptotically flat spacetime. Combining the proof of this result with arguments similar to above, it can be shown that if generalized ANEC holds along complete null geodesics γ in an asymptotically flat spacetime, with a microscopic (i. e., proportional to h) ANEC lower bound β_γ, then a nontrivial topological structure can be visible from infinity only if its spacetime curvature is of the Planck size.

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